

Semi-implicit two-step hybrid method with FSAL property for solving second-order ordinary differential equations



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ARTICLE INFO

Article history:

Received 3 March 2017

Received in revised form

16 May 2017

Accepted 27 May 2017

Keywords:

Hybrid method

Oscillatory solution

Interval periodicity

Interval of absolute stability

ABSTRACT

Two semi-implicit two-step hybrid methods of order five and six designed using First Same as Last (FSAL) property are developed for solving second-order ordinary differential equation. The stability analysis is determined by the interval of periodicity and the interval of absolute stability. The numerical results carried out show that the new method has smaller maximum error than existing method of similar type proposed in scientific literature, using constant step-size.

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1. Introduction

In this paper we are interested in the numerical solution of initial value problems (IVPs) associated with special second-order ordinary differential equation (ODE) of the forms (Eq. 1)

$$y'' = f(x, y), \quad y(0) = y_a, \quad y'(0) = y_b. \quad (1)$$

This problem does not incorporate the first derivative in $f(x, y)$ and the solution relates to oscillatory and periodic solutions. This type of problem commonly arises in the fields of applied sciences such as motion of planet in celestial mechanics, orbital problems, quantum mechanics and electronic. Since most differential equations of celestial mechanics take the form $y'' = f(x, y)$, it is not surprising that the first attempts at developing methods for Eq. 1 were made by astronomers (Hairer et al., 1993). In recent years, the special second-order ODEs have been extensively studied by many researchers for solving IVPs relates to oscillatory and periodic problems. There has been many research on multistep methods done for Eq. 1, particularly the two-step hybrid method (HM) (Tsitouras, 2003; Coleman, 2003; Franco, 2006; Fang and Wu, 2008; Samat et al., 2012; Ahmat et al., 2013; Jikantora et al., 2015; Franco et al., 2014; Franco and Rande, 2016; Kalogiratou et al., 2016). In a previous work, Coleman (2003), he has investigated the order

condition of two-step hybrid method based on the theory of B-series. He discussed these order conditions for general class of two-step hybrid method for problem Eq. 1.

A new class of explicit two-step hybrid method (EHM) which requires less number of stages per step has been developed by Franco (2006). He has considered EHM of order four up to order six. The study by Ahmad et al. (2013) developed semi-implicit two-step hybrid method up to algebraic order five for solving oscillatory problems by taking dispersion relation and solving them together with algebraic conditions of the methods. Later, another study carried out by Jikantora (2015) also developed semi-implicit two-step hybrid method with fifth algebraic order with dispersion and dissipation of higher order.

In this paper, we constructed semi-implicit two-step hybrid method with algebraic order five and six with FSAL feature. The FSAL feature specifies that, the last row of coefficient matrix is same with the vector of output value coefficients. The interval of stability for new methods are also presented and followed by numerical experiments on second-order differential equation for oscillatory or periodic problems. An s-stage two-step hybrid method generally given by (Eqs. 2 and 3)

$$Y_i = (1 + c_i)y_n - c_i y_{n-1} + h^2 \sum_{j=1}^s a_{ij} f(t_n + c_j h, Y_j) \quad (2)$$

$$y_{n+1} = 2y_n - y_{n-1} + h^2 [b_1 f_{n-1} + b_2 f_n + \sum_{i=3}^s b_i f(t_n + c_i h, Y_i)] \quad (3)$$

$$i = 1, \dots, s$$

The method consists of coefficients which is called the generating matrix, the vector output and

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the vector abscissa which can be represented in Butcher tableau as shown in Table 1.

Table 1: The coefficients of two-step hybrid method

c	A	c ₁	a _{1,1}	...	a _{1,s}
	b ^T	⋮	⋮	⋮	⋮
			c _s	a _{s,1}	...
			b ₁	...	b

2. Preliminaries

The method of the form Eq. 2 and Eq. 3 can be defined as Eqs. 4, 5, and 6.

$$Y_i = y_{n-1}, Y_2 = y_n, \tag{4}$$

$$Y_i = (1 + c_3) - c_3 y_{n-1} + h^2 \sum_{j=1}^i a_{ij} f(t_n + c_j h, Y_j) \tag{5}$$

= 3, ..., s

$$y_{n+1} = 2y_n - y_{n-1} + h^2 \left[\frac{b_1 f_{n-1} + b_2 f_n + \dots}{\sum_{i=3}^s b_i f(t_n + c_i h, Y_i)} \right] \tag{6}$$

where, $h_n = t_{n+1} - t_n$ is the step size while f_{n-1} and f_n represent approximations for $f(t_{n-1}, y_{n-1})$ and $f(t_n, y_n)$ respectively. The method only requires to evaluate $s - 1$ function evaluation namely $f(t_n, y_n), f(t_n + c_3 h, Y_3), \dots, f(t_n + c_s h, Y_s)$ in each step. Therefore, this method is considered as two-step hybrid method with $s - 1$ stages per step. The tableau of semi-implicit two-step hybrid method with FSAL (SIHMF) feature is as in Table 2

Table 2: The coefficients of SIHMF

-1	0				
0	0	0			
c ₃	a _{3,1}	a _{3,2}	γ		
⋮	⋮	⋮	⋮	γ	
c _s	a _{s,1}	a _{s,2}	⋮	a _{s,s-1}	γ
	b ₁	b ₂	⋮	b _{s-1}	γ

The diagonal elements $a_{33}, a_{44}, \dots, a_{ss}$ in Table 2 are denoted by γ . The vector output b^T corresponding to the output approximation is identical to the last row of A . The s -stage implicit two-step hybrid method parameters are given by

$$c_s = 1, a_{sj} = b_j$$

where, $j = 1, \dots, s$.

In this case, FSAL specifies the s -stage to be the same as the first-stage at the next step given by

$$f_s = f(x_{n+1}, y_{n+1}) = f(x_n + c_s h, (1 + c_s)y_n - y_{n-1} + \sum_{i=1}^s a_{sj} f_j) = f(x_n + h, 2y_n - y_{n-1} + \sum_{i=1}^s b_i f_i).$$

In order to investigate the phase property of two-step hybrid method for solving initial value problem Eq. 1 we consider the second order linear test equation as proposed by Franco (2006) (Eq. 7)

$$y'' = -\lambda^2(t) \tag{7}$$

If Eq. 2 and Eq. 3 are applied to the test problem Eq. 7, hence it can be written in the vector form as Eqs. 8 and 9

$$Y = (e + c)y_n - cy_{n-1} - H^2 AY \quad H = \lambda h, \tag{8}$$

$$y_{n+1} = 2y_n - y_{n-1} - H^2 b^T Y, \tag{9}$$

where $Y = (Y_1, \dots, Y_s)^T, c = (c_1, \dots, c_s)^T$ and $e = (1, \dots, 1)^T$. Solving equation in Eq. 8 we obtain Eq. 10

$$Y = (e + c)y_n(I + H^2 A)^{-1} - cy_{n-1}(I + H^2 A)^{-1}, \tag{10}$$

where

$$(I + H^2 A)^{-1} = 1 - H^2 A + H^4 A^2 = 1 - H^2 A + H^4 A^2 - \dots + (-1)^{s-2} H^{2s-4} A^{s-2}.$$

Substituting Eq. 10 in Eq. 9, then the following recursion relation is obtained Eq. 11:

$$y_{n+1} - S(H^2)y_n + P(H^2)y_{n-1} = 0, \tag{11}$$

where

$$S(H^2) = 2 - H^2 b^T (I + H^2 A)^{-1} (e + c) \text{ and } P(H^2) = 1 - H^2 b^T (I + H^2 A)^{-1} c$$

use $S(H^2)$, and $P(H^2)$ to define dispersion error and dissipation error.

Definition 1: The quantities $\varphi(H)$ and $d(H)$ are called the dispersion error (or phase-error) and dissipation error, respectively (Eq. 12)

$$\varphi(H) = H - \arccos\left(\frac{S(H^2)}{2\sqrt{P(H^2)}}\right) \tag{12}$$

$$d(H) = 1 - \sqrt{P(H^2)}.$$

According to Simos et al. (2003), the dispersion is the angle between the true and the approximate solution and the dissipation is the distance from a standard cyclic solution. The method is said to be dispersive of order q and dissipative of order r , if

$$\varphi(H) = (H^{q+1}) \quad d(H) = (H^{r+1})$$

the stability of two-step hybrid method will be calculated using the interval of periodicity and interval of absolute stability which are determined by characteristic polynomial.

Definition 2: The polynomial (Eq. 13)

$$\xi^2 - S(H^2)\xi + P(H^2) = 0 \tag{13}$$

is called characteristic polynomial of Eq. 11. Two-step hybrid method has periodicity interval if coefficient of Eq. 13 satisfy the condition (Eq. 14)

$$P(H^2) \equiv 1, |S(H^2)| < 2, \forall H \in (0, H_p) \tag{14}$$

The method satisfies the condition Eq. 14 are called zero dissipative ($d(H) = 0$). When the methods have a finite order of dissipation, means the interval of periodicity is $(0, \infty)$, the integration process is stable or remains bounded if the coefficient Eq. 13 satisfy the conditions (Eq. 15)

$$|P(H^2)| < 1,$$

and

$$|S(H^2)| < 1 + P(H^2) \quad (15)$$

$$\forall H \in (0, H_5)$$

The two-step hybrid method are derived by using order conditions which are the set of simultaneous equations which contain the coefficients. The solution of simultaneous equations gives the value of coefficients in terms of the free parameter associated to the local truncation error e_{p+1} . The coefficients are then substituted into the error constant E_{p+1} .

The minimized value of the free parameter is obtained by optimizing the error constant with respect to the free parameter. The p th-order error constant is a quantity defined by

$$E_{p+1} = \|e_{p+1}(t_1), \dots, e_{p+1}(t_k)\|_2$$

$$= \sqrt{e_{p+1}(t_1)^2 + \dots + e_{p+1}(t_k)^2}$$

where, k is the number of trees of order $p + 2(\rho(t_i) = \rho + 2)$ and $e_{p+1}(t_i)$ is local truncation error as defined in Coleman (2003). The order conditions up to algebraic order six given in Coleman (2003) are (Eqs. 17, 18, 19, 20, 21, 22)

Order 2
 $\sum b_i = 0,$ (17)

Order 3
 $\sum b_i c_i = 0$ (18)

Order 4
 $\sum b_i c_i^2 = \frac{1}{6},$ (19)

Order 5
 $\sum b_i c_i^3 = 0, \sum b_i a_{ij} c_j = 0$ (20)

Order 6
 $\sum b_i c_i^4 = \frac{1}{15}, \sum b_i c_i a_{ij} c_j = -\frac{1}{60}$ (21)

$$\sum b_i a_{ij} c_j^2 = \frac{1}{180}$$

and together with simplifying condition

$$\sum a_{3j} = \frac{c_3^2 + c_3}{2}, \sum a_{4j} = \frac{c_4^2 + c_4}{2} \quad (22)$$

3. Derivation of method SIHM with FSAL property

Presented in this section is the derivation of semi-implicit two-step hybrid method of order five designed using FSAL property.

3.1. Fifth order SIHM

In this case, FSAL specifies the four-stages to be the same as the first-stage at the next step given by

$$f_4 = f(x_{n+1}, y_{n+1}) = f(x_n + c_4 h, (1 + c_4)y_n - y_{n-1} + \sum_{i=1}^4 a_{4j} f_j) = f(x_n + h, 2y_n - y_{n-1} + \sum_{i=1}^4 b_i f_i)$$

The fourth-stage two-step hybrid method parameters are given by

$$c_4 = 1, \quad a_{4j} = b_i, \quad j = 1, 2, 3, 4$$

To derive fifth-order SIHMF method, we use the algebraic order conditions Eq. 17, Eq. 20 and Eq. 22. There are six equations and seven unknowns that have to be satisfied giving one free parameter which is chosen to be b_3 . The system of equations are solved simultaneously to obtain the values of coefficients in terms of b_3 which are given by the expression

$$b_1 = \frac{1}{12}, \quad b_2 = \frac{5}{6}, \quad c_3 = 1,$$

$$a_{31} = 0, \quad a_{32} = \frac{11}{12} + b_3, \quad a_{33} = -b_3 + \frac{1}{12}$$

by minimizing constant error in Eq. 16 we have

$$b_3 = 0$$

and

$$E_5 = 3.0619 \times 10^{-2}$$

This method is denoted as SIHMF5 and can be expressed diagrammatically as in Table 3. The interval of periodicity is given by $(0, \sqrt{6})$.

Table 3: The coefficients of SIHFM of order five

-1				
0				
1	0	$\frac{11}{12}$	$\frac{1}{12}$	
	$\frac{1}{12}$	$\frac{5}{6}$	0	$\frac{1}{12}$
1	$\frac{1}{12}$	$\frac{5}{6}$	0	$\frac{1}{12}$

3.2. Sixth order SIHM

In this case, FSAL specifies the five-stages to be the same as the first-stage at the next step given by

$$f_5 = f(x_{n+1}, y_{n+1}) = f(x_n + c_5 h, (1 + c_5)y_n - y_{n-1} + \sum_{i=1}^5 a_{5j} f_j) = f(x_n + h, 2y_n - y_{n-1} + \sum_{i=1}^5 b_i f_i).$$

The fifth-stage two-step hybrid method parameters are given by

$$c_5 = 1, \quad a_{5j} = b_i, \quad j = 1, 2, 3, 4, 5.$$

To derive the new method, we use the algebraic order conditions Eq. 17 and Eq. 22. There are 10 equations and 12 unknowns that have to be satisfied giving two free parameter which is chosen to be c_4 and γ . Solving all conditions simultaneously to obtain the values of coefficients in terms c_4 and γ . By minimizing constant error Eq. 4, we obtain $\gamma = -\frac{1}{40}$ and $c_4 = \frac{163}{100}$ and $E_6 = 4.4213 \times 10^{-2}$. This method is denoted as SIHMF6 and can be expressed diagrammatically as in Table 4. The interval of absolute stability is given by $(0, 2.18)$.

4. Results and discussion

In this section, we present five problems which have oscillatory solution. All the problems will be tested

by the semi-implicit FSAL methods to evaluate the

effectiveness of new method.

Table 4: The coefficients of SIHMF of order six

-1					
0					
473	1517188651	3679020971	1		
273	2441570040	1220785020	40		
163	2837920975353	9971265867353	25011725096631	1	
100	5968000000000	3784000000000	2822864000000000	40	
1	89909	401875	5554571841	130000000	1
	1177188	462594	19767105160	360228207	40
	89909	401875	5554571841	130000000	1
	1177188	462594	19767105160	360228207	40

The fifth order method, SIHMF5 is compared with semi-implicit hybrid method of order five with four-stages derived by Ahmad et al. (2013) and explicit two-step hybrid method of order four with three-stages derived by Franco (2006). The sixth order method is compared with two other methods derived by Franco (2006).

The methods that have been used in comparisons are denoted by

- (i) SIHM4(5): Fifth order semi-implicit method with four-stage derived by Ahmad et al. (2013).
- (ii) EHM4: Fourth order explicit hybrid method with four-stage derived by Franco (2006).
- (iii) EHM6: Sixth order explicit hybrid method with five-stage derived by Franco (2006).
- (iv) EHM5: Fifth order explicit hybrid method with four-stage derived by Franco (2006).

The criterion used in the numerical comparison is decimal logarithm of the maximum error versus step-sizes required by each method.

$$\text{Absolute error} = \max \text{error} |y(t_n) - y_n|$$

where, $y(t_n)$ is exact solution and y_n is approximate solution. The test problems used are listed below:

Problem 1: Homogeneous Problem studied by Franco (2006)

$$y'' = -y \quad y(0) = 0 \quad y'(0) = 1$$

Exact solution is $y = \sin(x)$. The numerical results are shown in Fig. 1 and Fig. 2.

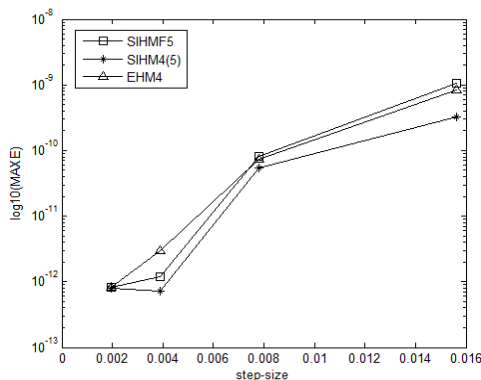


Fig. 1: The efficiency curve for SIHMF5 method for Problem 1 with $h = 0.125/2^i, i = 2,3,4,5$

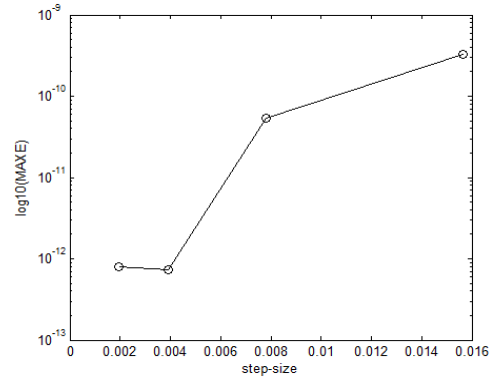


Fig. 2: The efficiency curve for SIHMF6 method for Problem 1 with $h = 0.125/2^i, i = 2,3,4,5$

Problem 2: The two-body gravitational problem studied by Dormand et al. (1987)

$$y_1'' = \frac{-y_1}{(\sqrt{y_1^2 + y_2^2})^3}, \quad y_1(0) = 1 - e, \quad y_1'(0) = 0,$$

$$y_2'' = \frac{-y_2}{(\sqrt{y_1^2 + y_2^2})^3}, \quad y_2(0) = 0, \quad y_2'(0) = \sqrt{\frac{1+e}{1-e}},$$

with e representing the eccentricity of an orbit. The exact solution is $y_1(x) = \cos(E) - e$ and $y_2 = \sqrt{1 - e^2} \sin(E)$ with E satisfies the Kepler's equation $x = E - e \sin(E)$. Numerical results is for the case $e = 0$. The numerical results are shown in Fig. 3 and Fig. 4.

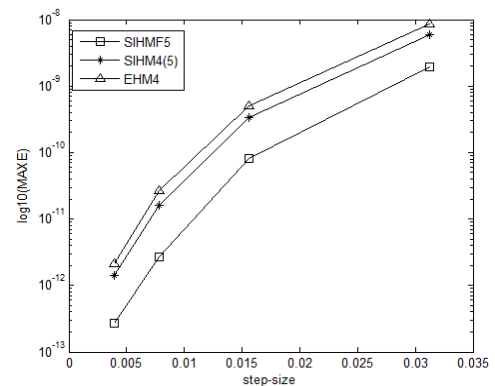


Fig. 3: The efficiency curve for SIHMF5 method for Problem 2 with $h = 0.125/2^i, i = 2,3,4,5$

Problem 3: Orbital Problem studied by Van der Houwen and Sommeijer (1989)

$$y_1'' = -4x^2 y_1 - \frac{2y_2}{\sqrt{y_1^2 + y_2^2}}, \quad y_1(0) = 1, \quad y_1'(0) = 0,$$

$$y_2'' = -4x^2 y_2 - \frac{2y_1}{\sqrt{y_1^2 + y_2^2}}, \quad y_2(0) = 0, \quad y_2'(0) = 0,$$

exact solution is $y_1(x) = \cos^2(x)$ and $y_2(x) = \sin^2(x)$. The numerical result are shown in Fig. 5 and Fig. 6.

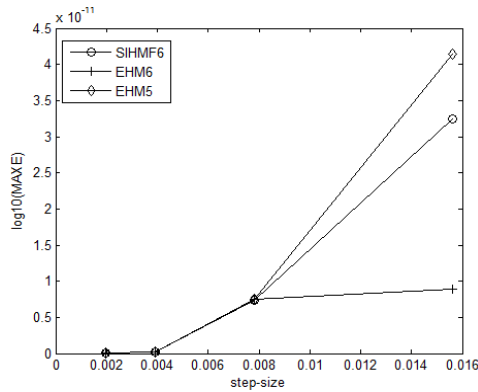


Fig. 4: The efficiency curve for SIHMF6 method for Problem 2 with $h = 0.125/2^i, i = 3,4,5,6$

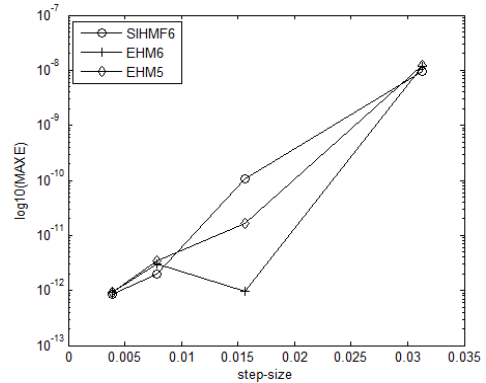


Fig. 6: The efficiency curve for SIHMF6 method for Problem 3 with $h = 0.125/2^i, i = 2,3,4,5$

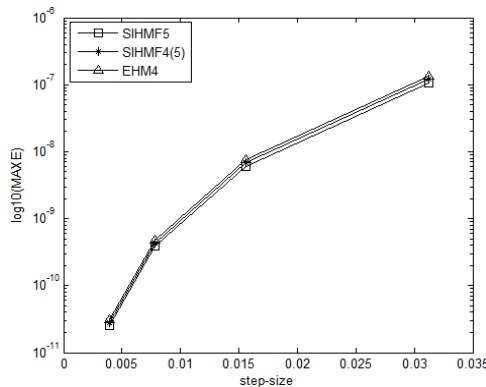


Fig. 5: The efficiency curve for SIHMF5 method for Problem 3 with $h = 0.125/2^i, i = 2,3,4,5$

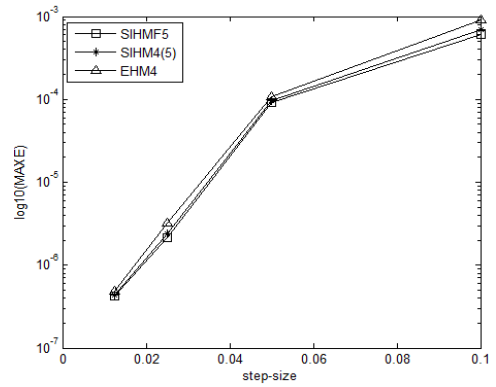


Fig. 7: The efficiency curve for SIHMF5 method for Problem 4 with $h = 0.1/2^i, i = 0,1,2,3$

Problem 4: Two-body problem studied by Franco (2003)

$$y_1'' + \omega^2 y_1 = \frac{2y_1 y_2 - \sin(2\omega x)}{\sqrt{(y_1^2 + y_2^2)^3}}, y_1(0) = 1, y_1'(0) = 0,$$

$$y_2'' + \omega^2 y_2 = \frac{y_1 - y_2 - \cos(2\omega x)}{\sqrt{(y_1^2 + y_2^2)^3}}, y_2(0) = 0, y_2'(0) = \omega$$

where $\omega = 1$ The exact solution is $y_1(x) = \cos(x)$ and $y_2(x) = \sin(x)$. The numerical results are shown in Fig. 7 and Fig. 8.

Problem 5: an almost periodic orbit problem given in Stiefel and Bettis (1969)

$$y_1'' = -y_1 + \cos(x), y_1(0) = 1, y_1'(0) = 0,$$

$$y_2'' = -y_2 + \sin(x), y_2(0) = 0, y_2'(0) = 0.9995,$$

exact solution is $y_1(x) = \cos(x) + 0.0005x \sin(x)$ and $y_2(x) = \sin(x) - 0.0005x \cos(x)$. The numerical results are shown in Fig. 9 and Fig. 10

From Fig. 3, Fig. 5 and Fig. 7, we observed that the new SIHMF5 is performed better compared to SIHMF4(5) and EHM4. However in Fig. 1 shows that SIHMF4(5) is performed better than SIHMF5 and EHM4. While in Fig. 9 we observed that, all methods have almost equal performance.

From Fig. 2, Fig. 4, Fig. 6, Fig. 8 and Fig. 10 we observed that the new SIHMF6 have almost equal performance with EHM6 and EHM5.

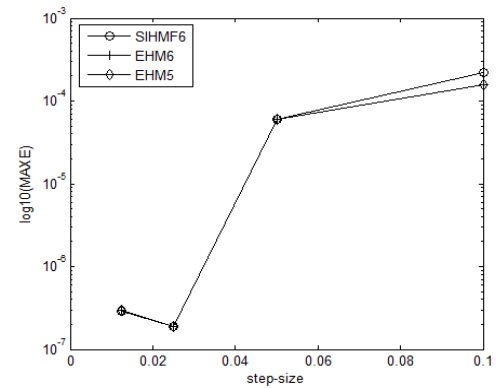


Fig. 8: The efficiency curve for SIHMF6 method for Problem 4 with $h = 0.1/2^i, i = 0,1,2,3$

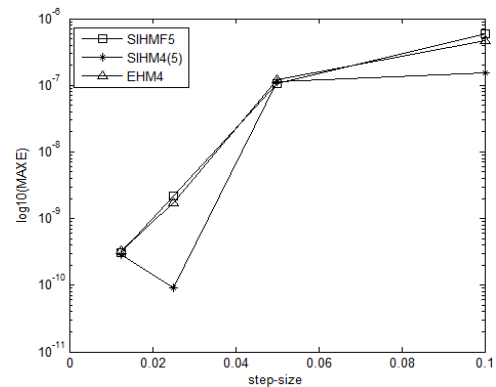


Fig. 9: The efficiency curve for SIHMF5 method for Problem 5 with $h = 0.1/2^i, i = 0,1,2,3$

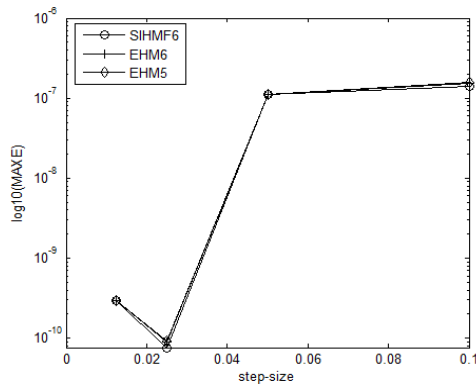


Fig. 10: The efficiency curve for SIHMF6 for Problem 5 with $h = 0.1/2^i, i = 0,1,2,3$

5. Conclusion

Two semi-implicit two-step hybrid method of order five and six designed using FSAL property for solving second-order IVPs with oscillatory solution are derived. The results of comparison based on maximum error evaluation at different step-sizes were used for comparison purpose as shown in Figs. 1–10. Our new method, SIHMF5 and SIHMF6 can be an alternative method for solving oscillatory problems and can be advantageous to the Science and Technology fields.

References

Ahmad SZ, Ismail F, and Senu N (2013). Semi implicit hybrid methods with higher order dispersion for solving oscillatory problems. In: Ishak A, Hashim I, Ismail ES, and Nazar R (Eds.), AIP Conference Proceedings, 1522(1): 553-560. <https://doi.org/10.1063/1.4801174>

Coleman JP (2003). Order condition for a class of two-step hybrid methods for $y'' = f(x, y)$. IMA Journal of Numerical Analysis, 23(2): 197 - 220.

Dormand JR, El-Mikkawy MEA, and Prince PJ (1987). Families of Runge-Kutta-Nystrom formulae. IMA Journal Numerical Analysis, 7(2): 235-250.

Fang Y and Wu X (2008). A trigonometrically fitted explicit numerov-type method for second order initial value problems with oscillating solution. Applied Numerical Mathematics, 58(3): 341-351.

Franco JM (2003). A 5 (3) pair of explicit ARKN methods for the numerical integration of perturbed oscillators. Journal of Computational and Applied Mathematics, 161(2): 283-293.

Franco JM (2006). A class of two-step hybrid methods for second-order IVPs. Journal of Computational and Applied Mathematics, 187(1): 41-57.

Franco JM, Gómez I, and Rández L (2014). Optimization of Explicit two-step hybrid methods for solving orbital and oscillatory Problems. Computer Physics Communications, 185(10): 2527-2537.

Franco JM, Rández L (2016). Explicit exponentially fitted two-step hybrid method of high order for second-order oscillatory IVPs. Applied Mathematics and Computation, 273: 493-505.

Hairer E, Norsett SP, and Wanner G (1993). Solving ordinary differential equations 1: Nonstiff Problems. Springer-Verlag Berlin Heidelberg, Berlin, Germany.

Jikantora YD, Ismail F, and Senu N (2015). Zero dissipative semi-implicit hybrid method for solving oscillatory or periodic problems. Applied Mathematics and Computation, 252: 388-396.

Kalogiratou Z, Monovasilis T, Higinio Ramos, and Simos TE (2016). A new approach on construction on trigonometrically fitted two-step hybrid methods. Journal of Computational and Applied Mathematics, 303: 146-155.

Samat F, Ismail F, and Suleiman M (2012). High order explicit hybrid methods for solving second order ordinary differential equations. Sains Malaysiana, 41(2): 253-260.

Simos TE, Famelis IT, and Tsitouras C (2003). Zero dissipative, explicit numerov-type methods for second order IVPs with oscillating solutions. Numerical Algorithms, 34(1): 27-40.

Stiefel E and Bettis DG (1969) Stabilization of Cowell's method. Numerische Mathematik, 13(2): 154-175.

Tsitouras C (2003). Families of explicit two-step methods for integration of problems with oscillating solutions. Applied Mathematics and Computation, 135(1): 169-178.

Van der Houwen PJ and Sommeijer BP (1989). Diagonally implicit Runge-Kutta-Nystrom methods for oscillatory problems. SIAM Journal on Numerical Analysis, 26(2): 414-429.